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Finite element methods as used in atmospheric models

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1. Simple finite element approximations

The finite element method is a way of representing data over some domain by using a discrete series of functions. The domain is divided into a finite number of subregions called elements. A series of functions is built up by defining a simple function, for instance a low order polynomial, on each element in turn and requiring continuity between functions on adjacent subregions. A simple example in one dimension is to divide a line into a series of segments and to define a function as linear on each segment and continuous between adjacent segments (Fig 1). In this case the function is completely defined by giving its values at the endpoints of the segments. In general we refer to points where values are used to define the function as nodes and the defining parameters as nodal values. Two more complicated examples are shown in Fig 2. Fig 2(a) shows a piecewise cubic function defined by the value and derivative at each end of the segments. The four parameters which are sufficient to define the cubic enable us to ensure that both the function and its derivative are continuous between elements. This scheme is usually called Hermite cubic interpolation. Fig 2(b) shows a two dimensional example. The function is defined as linear in each triangle, requiring three parameters to define it. These are the values at the three vertices. It is easy to see that this definition ensures continuity of the function between adjacent triangles.

Finite element approximations are distinguishable from spectral approximations in that they are not global but defined by local values. However, they are distinct from grid-point approximations because the function is defined over the whole region and not just at discrete points. Simple finite element ideas are often assumed when discussing grid-point methods. For instance, it is often assumed that functions vary linearly between grid points, or alternatively that grid-point values are mean values over "boxes".

A much more extensive description of finite element approximation is given in Strang and Fix (1973). We will only discuss a few important results here. The basic theorem states that if all polynomials up to degree r are used to construct the approximation over an element, then the approximation error is $O(h^{r+1})$ as h tends to zero where h is a typical element size. Similarly the pth derivative of a

function is approximated to order h^{r+1-p}. Thus piecewise linear approximation is second order accurate for functions and first order for derivatives. The Hermite cubics are fourth order accurate for functions and third order for derivatives. An important extension is that an elliptic equation of the form

$$L u = f \quad \text{on } \Omega \tag{1}$$

$$u \text{ or } \partial u/\partial n \quad \text{given on } \partial\Omega \text{ (the boundary of } \Omega \text{)}$$

can be solved to order h^{r+1} using polynomials up to degree r.

To illustrate what this result may mean in practice consider the approximation of a function expressed as a Fourier series $\int_{-\infty}^{\infty} a(k)\, e^{ikx}$. For synoptic scales in the atmosphere a -3 power law is approximately observed for the energy spectrum (Leith (1971)) so that the amplitude a(k) behaves like $|k|^{-3/2}$. Thus a spectral approximation representing the function by its first N Fourier compoents will have an error

$$\int_{N}^{\infty} + \int_{-\infty}^{-N} a(k)\, e^{ikx}\, dk$$

which is $O(N^{-\frac{1}{2}})$ for sufficiently large N. However, a finite element approximation using rth degree polynomials will have an error $O(N^{-(r+1)})$ and a grid-point representation an undefined error, if no assumption is made about the variation between grid-points. We see from this argument that the finite element method is a good way of representing functions. For reasons like this it is now a standard technique in engineering (Zienkiewicz (1971)).

2. Use of finite element approximations to solve differential equations

In this section we derive finite element approximations to some very simple differential equations of the same types as those met in atmospheric models, first of all steady-state problems such as those met in initialisation or which form part of the solution of balanced models; and then time dependent problems as met in forecasts. In steady-state problems it is easy to appreciate the way finite element methods behave. In time dependent problems it is more difficult, since we are usually more concerned with the rate of change of fields rather than in the representation

of the fields at a fixed time.

2.1 Solution of elliptic equations

Consider first the solution of the elliptic problem

$$Lu = f \quad \text{in a region } \Omega. \tag{2}$$

We seek to solve this in terms of a function $\tilde{u}$ which is defined by a finite element approximation. Substituting into (2) the error is

$$L\tilde{u} - f \equiv R \tag{3}$$

We have to set R=0 in some sense. Suppose that $\tilde{u}$ is defined by N parameters, so N equations are required to determine them. Possible choices are:

a. Set R=0 at N points in Ω (point collocation).

b. Set $\int_\Omega R\phi_n = 0$ for N functions ϕ_n (Galerkin method). The functions ϕ_n are called test functions.

c. Minimise $\int_\Omega (L\tilde{u} - f)^2$ over the N parameters (least squares).

Alternatively, if (2) can be written in variational form the finite element functions can be used as trial functions in a Rayleigh-Ritz procedure. Thus if the problem is

$$I(u) = \text{minimum over } \Omega$$

we minimise I with respect to the N parameters defining $\tilde{u}$.

In order to see how finite elements give a solution in practice write $\tilde{u}$ in the form $\sum u_n X_n$ where the values u_n are called nodal values and the X_n are functions on Ω called basis functions. Fig 3 illustrates the basis functions associated with the finite element approximations shown in Figs 1 and 2. For linear approximation the function X_n is piecewise linear, equal to 1 at the nth node and zero at all other nodes. In one dimension it is a "hat" function (Fig 3(a)) and in two dimensions a pyramid (Fig 3(d)). There are two types of basis function for the

Hermite cubics, one associated with the value of the function at the node (Fig 3(b)) and one associated with its first derivative (Fig.3(c)).

Now consider the finite element solution of the simple problem

$$\nabla^2 u = 0 \quad \text{on } \Omega, \quad u = f(x,y) \quad \text{on } \partial\Omega \tag{4}$$

Suppose for definiteness that Ω is a region in two-dimensional space with co-ordinates (x,y). Then the finite element representation of u takes the form $\Sigma u_n \chi_n(x,y)$. Consider solving the problem by the four techniques listed at the beginning of the section.

a. Point collocation This requires us to calculate $\nabla^2(\Sigma u_n \chi_n(x,y))$ at n points in terms of the u_n. This can only be done if $\chi_n(x,y)$ can be differentiated twice, and thus requires complicated functions.

b. Galerkin Using the basis functions as test functions we have to calculate integrals of the form

$$\int \nabla^2 \chi_n(x,y) \cdot \chi_m(x,y)\, d\Omega \tag{5}$$

Using integration by parts and enforcing the boundary condition u=0 on the basis functions this becomes

$$-\int \nabla\chi_n \cdot \nabla\chi_m \, d\Omega \tag{6}$$

This can be evaluated if the χ_n have first derivatives, and thus requires less complicated functions. The linear functions illustrated above would be sufficient. Consider a regular mesh with the numbering shown in Fig 3(d). Then $\nabla\chi_1 = \frac{1}{\Delta x}(-1, {}^2/\sqrt{3})$ in the triangle with vertices 1,2,4; $\frac{1}{\Delta x}(0, {}^2/\sqrt{3})$ in the triangle with vertices 1, 2, 3 and zero on the other triangles shown. By considering the product of $\nabla\chi_4$ with the other gradients in turn we obtain the

scheme

$$u_1 + u_2 + u_3 - 6u_4 + u_5 + u_6 + u_7 = 0 \qquad (7)$$

c. Least squares We have to minimise

$$\int \Sigma_n u_n \nabla^2 X_n \Sigma_m u_m \nabla^2 X_m \, d\Omega$$

by varying the u_n , giving the simultaneous equations

$$\int \nabla^2 X_n \cdot \Sigma_m u_m \nabla^2 X_m \, d\Omega = 0 \qquad (8)$$

These can only be solved if the X_n can be differentiated twice, and is therefore a more difficult procedure to apply than the Galerkin procedure.

d. Variational The variational principle for (4) is to minimise

$$\begin{aligned} I &= \int (\nabla u)^2 \, d\Omega \\ &= \int (\Sigma u_n \nabla X_n)^2 \, d\Omega \end{aligned} \qquad (9)$$

We therefore have to solve the simultaneous equations

$$\int \nabla X_n \cdot \Sigma u_m \nabla X_m \, d\Omega = 0$$

which are identical to those obtained from the Galerkin procedure and lead to (7) if the X_n are linear functions. However, the Galerkin procedure is more generally applicable since it does not depend on the existence of a variational principle.

The algorithm (7) obtained from the linear finite element approximation is very similar to the simple five-point finite difference representation of the Laplacian. It is true in general that the simplest finite element approximations lead to algorithms very similar to, or the same as, simple finite difference expressions. However, higher order finite element approximations do not usually give anything like the higher order finite difference schemes derived from Taylor expansions.

2.2 Solutions of evolutionary equations

Now consider using finite element methods to solve the hyperbolic problem

$$\frac{\partial u}{\partial t} = Lu \qquad (10)$$

as met in forecasts. It is usual to approximate u by an expression of the form $\sum u_n(t)X_n(x,y)$, for two space dimensions, leaving the time evolution to be computed by finite differences. It is clearly desirable to separate space and time variation in hyperbolic problems, e.g. by writing

$$\tilde{u} = \sum u_n \psi_n(t) X_n(x,y)$$

and in meteorological problems where the largest errors are believed to be due to space differencing it seems more profitable to use finite element representations in space only. We therefore solve (10) by setting

$$u_{t+\Delta t} - u_{t-\Delta t} = 2\Delta t\,(Lu)_t \equiv 2\Delta t\,\omega_t$$

where $\omega = \sum \omega_n X_n(x,y)$ is an expansion of the same form as $u = \sum u_n X_n(x,y)$ so that the addition $u_{t-\Delta t} + 2\Delta t\,\omega_t$ can be performed. Other finite-difference methods in time can be used in a similar way. The problem is therefore to calculate ω_n. As with the elliptic problem we define the remainder R as $(\omega - Lu)$ and set it equal to zero using the four methods considered in section 2.1.

In order to derive examples of the algorithms obtained using finite elements consider simple right-hand sides of (10), a first derivative $Lu = \partial u/\partial x$ and a product $L(u,v) = uv$. All the terms in the meteorological equations are combinations of these simple expressions. We first derive approximations to these using the linear one-dimensional finite element representation shown in Fig 1. Subsequently we consider higher order finite element approximations. Assume for simplicity that the u_n are values of u at equally spaced points $x = x_n$ with spacing Δx. A variable mesh spacing is considered later.

2.3 Linear finite element approximations to first derivatives

a. Point collocation Continuing with the notation of section 2.2, set $\omega = \frac{\partial u}{\partial x}$ at n points. If u is represented as a linear function then $\partial u/\partial x$ is piecewise constant over the intervals (x_n, x_{n+1}) and cannot be defined at the nodes x_n. However, since ω is required to be a linear function defined by its values at $x = x_n$ we are led to the system of simultaneous equations

$$\frac{1}{2}(\omega_n + \omega_{n+1}) = \frac{1}{\Delta x}(u_{n+1} - u_n) \qquad (11)$$

This system is singular and cannot be solved.

To use collocation at the nodes x_n we have to calculate $\partial u/\partial x$ at points where it is discontinuous. Using the average value

$$\frac{1}{2}\left\{ \left(\frac{\partial u}{\partial x}\right)_+ + \left(\frac{\partial u}{\partial x}\right)_- \right\}$$

we obtain the scheme

$$\omega_n = \frac{u_{n+1} - u_{n-1}}{2\Delta x} \qquad (12)$$

which is identical to the usual central difference approximation. However, in general, using higher order finite element interpolation with collocation does not give a higher order finite difference scheme.

b. Galerkin We set

$$\int (\omega - \partial u/\partial x)\, \chi_n = 0 \qquad (13)$$

using the basis functions as test functions. To evaluate (13) we have to calculate integrals of the form $\int \chi_m \chi_n \, d\Omega$. Using linear basis functions gives

$$\int \chi_{n+p} \chi_n = 0 \qquad |p| > 1$$

$$\int \chi_{n\pm1}\chi_n = \tfrac{1}{6}\Delta x$$

$$\int \chi_n^2 = \tfrac{2}{3}\Delta x \qquad (14)$$

$$\int \frac{\partial \chi_{n\pm1}}{\partial x}\chi_n = \pm\tfrac{1}{2}$$

$$\int \frac{\partial \chi_n}{\partial x}\chi_n = 0$$

Writing out $\omega = \Sigma_m \omega_m \chi_m$, $u = \Sigma_m u_m \chi_m$ and substituting the values for the integrals we find that

$$\tfrac{1}{6}\omega_{n-1} + \tfrac{2}{3}\omega_n + \tfrac{1}{6}\omega_{n+1} = \frac{1}{2\Delta x}(u_{n+1} - u_{n-1}) \qquad (15)$$

This is an implicit scheme for the first derivative which has been discussed in the literature on several occasions (e.g. Swartz and Wendroff (1974), Hirsch (1975)) since an error analysis gives its truncation error as $-\tfrac{1}{180}(\Delta x)^4$ which is less by a factor of 6 than the error of the explicit fourth order scheme

$$\omega_n = \frac{4}{3}\cdot\frac{1}{4\Delta x}(u_{n+2} - u_{n-2}) - \frac{1}{3}\cdot\frac{1}{2\Delta x}(u_{n+1} - u_{n-1}) \qquad (16)$$

However, the scheme (15) still gives a zero derivative if u is a two grid-length wave, as will any symmetrical scheme. Therefore the usual difficulties with reversed group velocities and stationary two grid-length "noise" are not resolved.

c. Least squares We minimise $\int(\omega - \partial u/\partial x)^2$ over the variables ω_n. This gives $\int \chi_n(\omega - \partial u/\partial x) = 0$ for all n. This is exactly the same as (13). In general, where the unknown field is not differentiated in space, the least squares and Galerkin methods will be the same.

d. Variational For the simple problem discussed here there is a variational principle which gives the same approximate scheme as the Galerkin method. For general fluid flow it is not usually possible to find a variational principle over the spatial domain, except the formal one obtained by minimising the squares of the equations.

At this point we note three further properties of the Galerkin or least squares method. It can be applied to an irregular mesh with no difficulty. Writing $h_{n+1/2} = x_{n+1} - x_n$ a little algebra gives

$$(1/6\,\omega_{n-1} + 1/3\,\omega_n)\,h_{n-1/2} + (1/3\,\omega_n + 1/6\,\omega_{n+1})\,h_{n+1/2} = 1/2\,(u_{n+1} - u_{n-1}) \qquad (17)$$

Analysis of this scheme, however, shows that it is only first order accurate. Practical tests with it on irregular grids show all the usual problems with information being unable to propagate out of a fine part of the mesh into a coarse part. The fourth order accuracy obtained with the scheme (15) is an accident. It is a case of "superconvergence", which is found to occur in many problems solved by numerical methods. Practical success with finite element methods seems to depend on taking advantage of it.

The second property is that the symmetry of the inner products $\int X_n X_{n\pm p}$ forces the resulting scheme (15) to be symmetrical. This would indicate a lack of flexibility in comparison with finite differences. However, it is possible to avoid this restriction by using test functions ϕ_m different from the functions X_m used to derive the finite element interpolation. A very simple example is to consider the test functions

$$\phi_n(x) = \frac{x_n - x_{n-1}}{x_n - x_{n-1}} \qquad x_{n-1} \le x \le x_{n+1}$$

$$= 0 \qquad \text{elsewhere}$$

The scheme obtained using these is

$$1/6\,\omega_{n-1} + 1/3\,\omega_n = 1/2\Delta x\,(u_n - u_{n-1}) \qquad (18)$$

which is a difference approximation weighted towards a one-sided difference. This has been recently explored by Zienkiewicz for engineering applications. The third property is that of satisfying conservation laws. Multiply the nth member of the system (13) by u_n and sum over n giving

$$\int \left(\omega - \partial u/\partial x\right) \Sigma u_n \chi_n = 0 \qquad (19)$$

so that $\int (\omega - \partial u/\partial x) u = 0$ exactly. In many problems this equation expresses conservation of energy or momentum. In general any conservation law which can be expressed by multiplying the differential equation by one variable and integrating will be satisfied exactly in the Galerkin scheme. This is not necessarily a good thing, for instance Raymond and Garder (1976) find conservation to be a disadvantage on irregular grids.

2.4 Approximation of products

Now consider the second example, $L(u,v) = uv$ the simplest possible non-linear expression. We will only consider the collocation and Galerkin methods.

a. Point collocation Using collocation at the points x_n gives

$$\omega_n = u_n v_n \qquad (20)$$

In the particular case $u = v = +1$ (n even), $= -1$ (n odd), this gives $\omega_n = 1$ for all n. (See Fig 4). This is the sort of scheme that leads to aliasing (Vol.I.p.35) and non-linear instability.

b. Galerkin Using the basis functions as test functions we have to solve

$$\int (\omega - uv) \chi_n = 0 \qquad (21)$$

The integrals required are

$$\int X_{n\pm1}^2 X_n = \int X_{n\pm1} X_n^2 = 1/12$$

$$\int X_n^3 = 1/2$$

$$\int X_{n+1} X_n X_{n-1} = 0$$

The scheme is

$$1/6\,\omega_{n-1} + 2/3\,\omega_n + 1/6\,\omega_{n+1} = 1/12\,(u_{n-1}v_{n-1} + u_{n-1}v_n + u_n v_{n-1}) + 1/2\,u_n v_n + 1/12\,(u_n v_{n+1} + u_{n+1}v_n + u_{n+1}v_{n+1}) \qquad (22)$$

Applying this scheme to the two grid length wave case $u = v = \pm 1$ gives $\omega_n = 1/3$ for all n, (Fig.4). Fourier analysis of this scheme, which is done in detail in section 3, shows that (22) damps short wave interactions.

The difference between schemes (20) and (22) is related to the question of aliasing in finite difference schemes. If grid point values are regarded purely as values at that point, then there is no error in scheme (20). However, this assumption is not reasonable in a case where information propagates from point to point. Therefore we must interpret the grid-point values either as defining Fourier components or as defining something like a piecewise linear function. Then the grid-point values of ω defined by (20) will not be the best values to define uv in general. For the piecewise linear assumption the scheme (22) is the most accurate method being a best least-squares fit, for the Fourier assumption we have to truncate at an arbitrary wavenumber. In a numerical method where a function is defined over the whole domain, by finite elements or other means, there can be no aliasing since there is no ambiguity. However there is still truncation error because the square of a piecewise polynomial is a polynomial of higher order which has to be approximated by a lower order one. This error will be analysed in section 3.

We can achieve greater economy in using (22) by rewriting it in the form

$$\tfrac{1}{6}\omega_{n-1} + \tfrac{2}{3}\omega_n + \tfrac{1}{6}\omega_{n+1} = \tfrac{1}{3}\left(\frac{u_n+u_{n+1}}{2}\right)\left(\frac{v_n+v_{n+1}}{2}\right) + \tfrac{1}{3}u_n v_n$$
$$+ \tfrac{1}{3}\left(\frac{u_n+u_{n-1}}{2}\right)\left(\frac{v_n+v_{n-1}}{2}\right)$$

This means that the product approximation can be built up by calculating a simple product on a finer mesh $(n-1, n-\tfrac{1}{2}, n, n+\tfrac{1}{2}, n+1)$ than the original one and taking a weighted mean over this mesh. This is exactly analogous to what happens in the spectral transform technique where spherical harmonics are transformed onto a grid, the products calculated on this grid, and then transformed back. However, in the case of (22), the time saved is only marginal.

2.5 Approximation of non-linear advective terms

Next consider the evaluation of a non-linear advective term of the form $\omega = u\frac{\partial v}{\partial x}$. In finite difference calculations it is found that care must be taken to prevent non-linear instability (Vol.I p.35). Thus the simple scheme

$$\omega_n = \frac{1}{2\Delta x} u_n (v_{n+1} - v_{n-1}) \qquad (23)$$

is found to be unstable and the alternative scheme

$$\omega_n = \left\{\frac{1}{2\Delta x}(u_n + u_{n-1})(v_n - v_{n-1}) + \frac{1}{2\Delta x}(u_n + u_{n+1})(v_{n+1} - v_n)\right\}/2 \qquad (24)$$

used instead. It is interesting to see what kind of scheme can be produced using finite element approximations. We consider various possible methods using linear basis functions.

a. Point collocation This suffers from the same difficulty as in the linear case, the derivative is naturally calculated at the midpoint of the element, while values of w are needed at the nodes. In order to get equations for w that can be solved we have to estimate the derivative at the node, giving

$$\omega_n = \tfrac{1}{2\Delta x} u_n (v_{n+1} - v_{n-1})$$

exactly as (23). However, the right hand side of the equation at the midpoint of the element is exactly one of the terms on the right hand side of (24),

the left-hand side contains terms like $\frac{1}{2}(\omega_n + \omega_{n+1})$ which cannot be inverted.

b. Galerkin Set $\int(\omega - u\frac{\partial v}{\partial x})\chi_n = 0$ for all the test functions. We have to evaluate integrals like $\int \chi_m \frac{\partial \chi_l}{\partial x} \chi_n$ and after some algebra we obtain the scheme

$$\tfrac{1}{6}\omega_{n-1} + \tfrac{2}{3}\omega_n + \tfrac{1}{6}\omega_{n+1} =$$

$$\tfrac{1}{2\Delta x}\{(\tfrac{1}{3}u_{n-1} + \tfrac{2}{3}u_n)(v_n - v_{n-1}) + (\tfrac{1}{3}u_{n+1} + \tfrac{2}{3}u_n)(v_{n+1} - v_n)\} \qquad (25)$$

Since this is a Galerkin scheme it is conservative as we can write $\int \phi(\omega - u\,\partial v/\partial x) = 0$ for any variable ϕ. However, it is shown in section 3 that the error made by it is quite large, about $\frac{1}{40}(\Delta x)^4$. The method is illustrated in Fig 5. The problem is that $\partial v/\partial x$ is represented by a series of constants, which is unsatisfactory near the nodes. When this profile is multiplied by a non-constant u the inaccuracies of $\partial v/\partial x$ can lead to serious inaccuracies in the product. This suggests an alternative approach:

(26)

(i) Calculate $s = \partial v/\partial x$ using (15)

(ii) Calculate the product $\omega = us$ using (22)

This method is found to be more accurate in practice than (25). It has errors of about $(\Delta x^4)/240$. However, it no longer conserves "energy"; (Cullen (1974a)). Therefore there is now a direct choice between accuracy and conservation; while with the simple finite difference schemes (23) and (24) cnnservation was essential to obtain stability. It is likely that this is because the aliasing effect is removed in the finite element integrations and that instability is made less likely. This scheme is also illustrated in Fig 5.

2.6 Use of higher order finite element schemes

Now consider one example of using a higher order finite element approximation, the quadratic on line segments. There are two sorts of basis function, illustrated in Fig 6. The nodal values are values at the endpoints and the midpoints of the segments. One type of basis function is zero at the endpoints of each element, and

unity at the midpoint. This function is only non-zero on one element. The other type is non-zero on two elements and is zero at all midpoints and all except one endpoint. The formulae for these functions are:

a) $1 + 3x + 2x^2 \qquad -1 \le x \le 0$

$1 - 3x + 2x^2 \qquad 0 \le x \le 1$

b) $4x - 4x^2 \qquad 0 \le x \le 1$

Consider evaluating $\omega = \partial u/\partial x$ using this representation:

a. Point collocation At endpoints of segments we get, using the average derivative at the endpoint:

$$\omega_0 = \tfrac{1}{2\Delta x}(-u_1 + 4u_{1/2} - 3u_0 + u_{-1} - 4u_{-1/2} + 3u_0)$$

$$= \tfrac{1}{2\Delta x}(4(u_{1/2} - u_{-1/2}) + u_1 - u_{-1}) \qquad (27)$$

At midpoints we get

$$\omega_{1/2} = \tfrac{1}{\Delta x}(u_1 - u_0)$$

These are two different explicit schemes for the first derivative. Others can be obtained by collocation at different points within the elements. Both these schemes are second order accurate, by collocation at suitable special points we can make up a fourth order scheme.

b. Galerkin After considerable algebra the same techniques as were used for linear elements give:

$$\omega_0 + 8\omega_{1/2} + \omega_1 = \tfrac{10}{\Delta x}(u_1 - u_0)$$

$$-\omega_{-1} + 2\omega_{-1/2} + 8\omega_0 + 2\omega_{1/2} - \omega_1 = \tfrac{10}{\Delta x}(2(u_{1/2} - u_{-1/2}) - \tfrac{1}{2}(u_1 - u_{-1})) \qquad (28)$$

This is a pair of implicit schemes, both of second order accuracy. This time the use of the Galerkin method does not gain accuracy.

Normally, when Galerkin methods are used with higher order finite elements, we do not evaluate the integrals $\int \chi_n \chi_n$, $\int \chi_m \frac{\partial \chi_n}{\partial x}$ exactly. They are evaluated by numerical integration. Consider the effect of using Simpson's rule to solve $\omega = \partial u / \partial x$ by the Galerkin method using quadratic functions. The two equations are:

$$\text{(i)} \quad \int_{-1}^{1} \chi_0 \left(u_0 \frac{\partial \chi_0}{\partial x} + u_{1/2} \frac{\partial \chi_{1/2}}{\partial x} + u_{-1/2} \frac{\partial \chi_{-1/2}}{\partial x} + u_1 \frac{\partial \chi_1}{\partial x} + u_{-1} \frac{\partial \chi_{-1}}{\partial x} \right)$$

$$= \int_{-1}^{1} \chi_0 \left(\omega_0 \chi_0 + \omega_{1/2} \chi_{1/2} + \omega_{-1/2} \chi_{-1/2} + \omega_1 \chi_1 + \omega_{-1} \chi_{-1} \right)$$

Evaluating these integrals as $\frac{1}{12} \{ \underline{I}_{-1} + 4\underline{I}_{-1/2} + 2\underline{I}_0 + 4\underline{I}_{1/2} + \underline{I}_1 \}$ we obtain

$$1 \cdot (0 + 4u_{1/2} - 4u_{-1/2} - u_1 + u_{-1}) = \omega_0$$

since $\chi_0 = 0$ at $x = \pm 1/2, \pm 1$ by definition.

$$\text{(ii)} \quad \int_0^1 \chi_{1/2} \left(u_0 \frac{\partial \chi_0}{\partial x} + u_{1/2} \frac{\partial \chi_{1/2}}{\partial x} + u_1 \frac{\partial \chi_1}{\partial x} \right) = \int_0^1 \chi_{1/2} \left(\omega_0 \chi_0 + \omega_{1/2} \chi_{1/2} + \omega_1 \chi_1 \right)$$

Evaluating these integrals as $\frac{1}{6}(\underline{I}_0 + 4\underline{I}_{1/2} + \underline{I}_1)$ we obtain

$$u_1 - u_{-1} = \omega_{1/2}$$

since $\chi_{1/2} = 0$ at $x = \pm 1$. Therefore the algorithm is exactly the same as that obtained by point collocation (27).

The property that we obtain two different schemes, one for endpoints and one for midpoints, is general with higher order elements. In general there are two wave speeds instead of one. For a correct choice of initial data we may only get one solution in a linear problem and this will be satisfactory. However, in a non-linear problem the other solution will be excited and this can cause difficulty, it may be necessary to discard the midpoint values and replace them with values interpolated from the endpoints. If this has to be done there is no point in using quadratics at all. We also note that the advection scheme obtained from the quadratics is less accurate asymptotically than that obtained from linear functions.

2.7 Summary

In general it has been shown that applying finite element methods to hyperbolic problems can either lead to the same simple explicit schemes as usually used from finite difference arguments, or to implicit schemes which may be much more accurate for advection and remove the aliasing problem. However, higher order finite element schemes do not necessarily give better approximation schemes for these problems. The reason for this is discussed in the next section. For non-linear problems time can be saved by a technique parallel to the spectral transform technique for evaluating products, and simpler schemes can be derived from higher order Galerkin schemes by using numerical integration.

3. Analysis of the finite element method applied to differential equations.

3.1 Introduction

We now illustrate the points made about different finite element methods by carrying out a formal error analysis. The basic result, described by Strang and Fix (1973), is that using polynomial interpolation with all polynomials up to degree r gives an approximation error $O(h^{r+1})$ where h is a typical element size. This result can be extended to the solution of elliptic equations. However, it does not apply to the solution of first order hyperbolic equations using the Galerkin method. Lesaint (1975) shows that an error $O(h^r)$ is obtained in this case, though there are many circumstances in which $O(h^{r+1})$ can be recovered. However, in solving evolutionary problems it is more important to know how well a method approximates wave speeds, for instance, than how well it approximates the shape of the initial data. We therefore perform an error analysis that reflects this.

3.2 General principles of error analysis for evolutionary problems

Consider the equation $\frac{\partial u}{\partial t} = Lu$. Suppose that initially $u = u_0$ and we wish to integrate forward in time. Then the exact solution is $u = u_0 + \int Lu\, dt$ In order to get a numerical solution, by finite elements or any other method, we first have to approximate u_0 by grid-point or nodal values. This approximation is represented by an operator p, so that the discrete initial data is written as pu_0 The operator L must be replaced by an operator $\tilde{L}$ acting on discrete values. In

the case of the Galerkin method $\tilde{L}$ is simply pL as we can see because we define p by setting $\int(\rho u_0 - u_0)X_n = 0$ and $\tilde{L}$ by $\int(\tilde{L}(\rho u) - L(\rho u))X_n = 0$. The time integration is replaced by a finite difference, replacing $\int dt$ by the summation $\sum dt$. Thus the numerical solution to the problem can be written as $\rho u_0 + \sum \tilde{L}(\rho u)\Delta t$. This has to be compared with the discrete values derived from the exact solution; $\rho(u_0 + \int L u\, dt)$; we cannot compare discrete values with a continuous solution directly. The error is thus

$$\sum \tilde{L}(\rho u)\Delta t - \rho \int L u\, dt \qquad (29)$$

it is illustrated in Fig 7.

To clarify how this error is built up write A for the operator advancing the true solution from t to $t+\Delta t$ and B for the operator advancing the numerical solution fromlt to $t+\Delta t$ Then the error (29) can be written as

$$\begin{aligned} & \rho A^n u_0 - B^n \rho u_0 \\ &= (\rho A^n - B\rho A^{n-1})u_0 + (B\rho A^{n-1} - B^2 \rho A^{n-2})u_0 + \cdots \\ &\quad + (B^{n-1}\rho A - B^n \rho)u_0 \end{aligned} \qquad (30)$$

Each term of (30) can be written as

$$(B^{n-k}\rho A^k - B^{n-k}B\rho A^{k-1})u_0 \equiv (B^{n-k}\cdot\rho A - B^{n-k}\cdot B\rho)A^{k-1}u_0 \qquad (31)$$

The factor B^{n-k} cannot be taken out from (31) since B may be non-linear. This expansion shows that the total error can be estimated if we can estimate

(i) The new error at each step $\rho A - B\rho$

(ii) The accumulation of error due to errors at previous steps, i.e. given $\|u - v\| \leq \epsilon$ we have to bound $\|B^n u - B^n v\|$

Requirement (ii) involves both the stability of the numerical scheme and the predictability of the operator B. In fact we will want $\|B^n u - B^n v\|$ to grow rapidly if $\|A^n u - A^n v\|$ does so. The error (i) is just a generalisation of the

truncation error calculated for finite difference schemes. In finite differences the projection p is simply the selection of grid-point values. The error is the difference between the exact solution at the grid-point and the numerical solution there. This generalised analysis allows us to consider finite element and spectral methods as well. In the next section we analyse the approximations to derivatives and products derived in section 2 and also perform some stability analyses. However, a complete estimation of the error (31) depends on a knowledge of the behaviour of the non-linear problem being solved, and is usually not possible.

3.2 Error analysis for linear finite element approximations on a regular mesh.

We first of all analyse the error for linear finite element approximations in one dimension on a regular mesh with periodic boundary conditions. Assume that $\Delta x = 1$. The Fourier method can be used for the error analysis in this case. We analyse the behaviour of a single component $u = Re\,(e^{ikx})$. First of all, consider the effect of the projection p. This is illustrated in Fig 8. We have to represent u by a piecewise linear function. This is most naturally done by a least-squares fit; as shown in curve A. However, it could also be done by reading off the values of u at the nodes and interpolating, giving curve B. The first method is a Galerkin approximation to u, the second is a point collocation. Some algebra shows that the nodal values of u defining curves A and B are $a(k)e^{ikx_n}$ where

$$a(k) = \frac{12}{4+2\cos k}\left(\frac{1-\cos k}{k^2}\right) \qquad \text{for A}$$

$$= 1 \qquad \text{for B} \qquad (32)$$

The maximum approximation error $pu-u$ is $O(k^4)$ as k tends to zero for either case. This illustrates the general result that the approximation error is $O(k^4)$ for polynomials of degree 1.

Now consider the error in approximating $w = Lu = \partial u/\partial x$. The collocation scheme (12) gives

$$w_n = \tfrac{1}{2}(u_{n+1} - u_{n-1})$$

Given $u=e^{ikx}$ we derive each term in the error (29) as follows:

$$pu = e^{ikx_n}$$

$$Lu = ike^{ikx}$$

$$pLu = ike^{ikx_n}$$

$$\tilde{L}(pu) = \tfrac{1}{2} e^{ikx_n}(e^{ik\Delta x} - e^{-ik\Delta x}) = i\sin k\, e^{ikx_n} \quad , \text{ as } \Delta x = 1$$

The error (29) is therefore

$$\tilde{L}(pu) - pLu = i(\sin k - k)e^{ikx_n} \tag{33}$$

which is $O(k^3)$ as $k \to 0$. This analysis exactly parallels the finite difference truncation error analysis of the same scheme.

Consider next the Galerkin scheme. Derive the initial data from (32) and use (15) to approximate the derivative so that

$$\tfrac{1}{6}\omega_{n-1} + \tfrac{2}{3}\omega_n + \tfrac{1}{6}\omega_{n+1} = \tfrac{1}{2}(u_{n+1} - u_{n-1})$$

Given $u = e^{ikx}$ we can write

$$pu = a(k)e^{ikx_n}$$

$$Lu = ike^{ikx}$$

$$pLu = ika(k)e^{ikx_n}$$

$$\tilde{L}(pu) = \frac{6}{4+2\cos k} \cdot i\sin k\, a(k)e^{ikx_n}$$

The error is therefore $\tilde{L}(pu) - pLu =$

$$ia(k)e^{ikx_n}\left(\frac{6\sin k}{4+2\cos k} - k\right) \tag{34}$$

which is $O(k^4)$ as $k \to 0$.

There are several things to note about this result. The increased wave speeds mean that the maximum time step that can be taken with explicit time differencing is reduced by $\sqrt{3}$, compared to reductions of $4/3$ for explicit fourth order differencing and π for the spectral method. The projection factor $\alpha(h)$ comes out as a factor so that the scheme would still be fourth order accurate if it was used with interpolated initial data. This is the way that it is analysed by Swartz and Wendroff (1974). The order of accuracy is much higher than would be expected from standard finite element theory. In general we would only expect first order accuracy using linear functions. The higher accuracy comes because $\tilde{L}\rho u$ almost cancels $\rho L u$. This turns out to be a general property of Galerkin schemes using spline functions on a regular mesh (Thomée and Wendroff (1974)) and is not true for the quadratic functions discussed in section 2.6 (A spline is a piecewise polynomial of degree r with r - 1 continuous derivatives between elements). The general requirement for high accuracy seems to be that all the nodal values are associated with the same discrete equation, e.g. (15), for approximating L. Thus the quadratic scheme where there are two kinds of discrete equation (as (28)) or schemes on irregular grids (17) where the discrete equation will be different at every node do not give high accuracy but only the basic second and first order accuracy respectively expected from the order of polynomial used.

3.3 Error analysis for higher order schemes and irregular meshes.

We follow up the statements of 3.2 by carrying out the analysis for the quadratic case (28) and the irregular mesh (17). The analysis of (28) is complicated because the wave e^{ikx} has to be fitted by two types of polynomial based on the endpoints and the midpoints. Write

$$(\rho u)_n = \alpha(k) e^{ikx_n}$$

$$(\rho u)_{n+1/2} = \beta(k) e^{ikx_n} \qquad (35)$$

Then

$$(\rho L u)_n = ik\alpha(k) e^{ikx_n}$$

$$(\rho L u)_{n+1/2} = ik\beta(k) e^{ikx_n}$$

Write

$$(\tilde{L}\rho u)_n = \gamma(k)\alpha(k) e^{ikx_n}$$

$$(\tilde{L}\rho u)_{n+1/2} = \delta(k)\beta(k) e^{ikx_n}$$

Substituting (35) into (28) and expanding in powers of k gives

$$2\gamma\alpha(1-k^2/8+\cdots)+\delta\beta = 2\alpha(k/2-k^2/48+\cdots)$$

$$\gamma\alpha(6+k^2+\cdots)+4\delta\beta(1-k^2/8+\cdots) = 4\alpha(k/2-k^2/48+\cdots)\beta - 10(k-k^3/6+\cdots)\alpha \qquad (36)$$

We know that the quadratics will approximate $u=e^{ikx}$ to third order accuracy, so that $\alpha(k)=1+O(k^3)$, $\beta(k)=1+O(k^3)$. In addition derivatives will be approximated at least to second order, so that $\gamma(k)=ik(1+O(k^2))$ $\delta(k)=ik(1+O(k^2))$. Substituting into (36) and considering terms in k^3 shows that the second order errors a and b in the derivatives are non-zero, satisfying

$$2a+8b=-1/6$$
$$6a+4b=1/3$$

Therefore (28) is only second order accurate. In addition there are two values of the wave speed for the given wave e^{ikx}, this will lend to a distortion of the wave representation (35) into a shape that can be advected at constant speed. This scheme has been used with reasonable success in a linear problem by Gresho et al (1976).

The Fourier method cannot be used to analyse (17), because of the irregular mesh. Instead we expand the functions in Taylor series. We recall the scheme

$$(1/6\,\omega_{n-1}+1/3\,\omega_n)h_{n-1/2}+(1/3\,\omega_n+1/6\,\omega_{n+1})h_{n+1/2} = 1/2(u_{n+1}-u_{n-1})$$

Expand in a Taylor series about x_n. Then we obtain

$$-1/6\,h^2_{n-1/2}\omega' + 1/2\,\omega_n(h_{n-1/2}+h_{n+1/2}) + 1/6\,h^2_{n+1/2}\omega' + \cdots$$
$$= 1/2(h_{n+1/2}u' + 1/2\,h^2_{n+1/2}u'' + h_{n-1/2}u' - 1/2\,h^2_{n-1/2}u'' + \cdots)$$
$$\therefore\ 1/2\,\omega_n(h_{n-1/2}+h_{n+1/2}) + 1/6\,\omega'(h^2_{n+1/2}-h^2_{n-1/2}) + \cdots =$$
$$1/2\,u'(h_{n-1/2}+h_{n+1/2}) + 1/4\,u''(h_{n+1/2}-h_{n-1/2})(h_{n-1/2}+h_{n+1/2}) + \cdots$$

Thus to zero order $\omega=u'$ as desired; but the first order terms do not balance. Therefore (17) is only first order accurate.

In summary, the analysis of the last two sections shows that very accurate schemes can sometimes be obtained using finite elements. However in general

only the standard $O(h^r)$ can be obtained. The error of the accurate scheme (15) is illustrated in Fig 9 in comparison with the simple centred second and fourth order finite difference schemes (12) and (16).

3.4 Error analysis of non-linear problems

Now consider non-linear problems. The projection factor $\alpha(k)$ will no longer come out as a factor in the error analysis and therefore the result of the analysis will be affected by the choice of projection. We illustrate this by analysing the schemes (20) and (22) assuming first of all that the initial data is interpolated and then that the initial data is generated by a least squares fit with $\alpha(k)$ given by (32). This sort of distinction is exactly parallel to the finite difference case where, considering only values at grid-points, a simple product (20), has no error, while, re-interpreting the data as Fourier components, there is an error.

Consider a single non-linear interaction $w=uv$ with $u=Re(e^{ikx})$ and $v=Re(e^{ilx})$. Then the discrete values u_n and v_n are given by

$$u_n = \alpha(k)e^{ikx_n}, \quad v_n = \alpha(l)e^{ilx_n}$$

By periodicity the discrete values of w will be given by

$$w_n = \beta(k,l)\,\alpha(k)\,\alpha(l)\,e^{i(k+l)x_n}$$

where $\beta(k,l)$ depends on the algorithm used. For (20) $\beta(k,l)=1$
For (22) Fourier analysis gives

$$\beta(k,l) = \tfrac{1}{6}(3+\cos k+\cos l+\cos(k+l)) \qquad (37)$$

The exact solution for w is given by

$$w_n = \alpha(k+l)\,e^{i(k+l)x_n}$$

so that the error is proportional to

$$\alpha(k+l) - \alpha(k)\,\alpha(l)\,\beta(k,l) \qquad (38)$$

The errors using (20) and (22) are summarised in the table below.

Scheme	(20)	(22)
Initial Data		
Interpolation	0	$\frac{1}{12}(k^2+l^2+(k+l)^2)+O(\|k\|+\|l\|)^4$
Fitting	$\frac{1}{6}kl+O(\|k\|+\|l\|)^4$	$-\frac{1}{720}(2k^3l+3k^2l^2+2kl^3)+O(\|k\|+\|l\|)^6$

As might be expected the most accurate schemes are the natural pairings of (20) with interpolated data and (22) with fitted data. To choose between the pairs consider the short waves with k and l large. For interpolated data $\alpha(k)=1$ for all k but for fitted data $\alpha(k)=O(1/k^2)$. Therefore fitting damps out the high wave number noise. The associated product algorithm (22) damps out shortwave interactions. Thus in practice we can hope that using (22) with fitting will be more stable than using interpolation with (20), in exactly the same way that using a spectral truncation with no aliasing is more stable than a simple product. This is found to be true in practice (Cullen (1974a)) and it is possible to produce more rigorous arguments to justify it.

We finally analyse the schemes for non-linear advective terms set out in section 2.5. Using the Fourier method and substituting $u=Re(e^{ikx})$, $v=Re(e^{ilx})$ gives

$$\beta(k,l)=\frac{1}{4+2\cos(k+l)}(4\sin l+2\sin(k+l)-2\sin k)$$

The error, (38), is then

$$il(2k^3l-7k^2l^2-8kl^3-4l^4)/720+O(|k|+|l|)^6 \qquad (39)$$

using the values (32) for $\alpha(k)$. Using the two-stage scheme (26) the error is the product of the derivative error (34) and the product error (37). This gives

$$il(2k^3l+3k^2l^2+2kl^3-l^4)/720+O(|k|+|l|)^6 \qquad (40)$$

The error (40) is usually less than (39) so that the two stage scheme should be superior. This was found by Cullen (1974a). If (25) and (26) are analysed without allowing for the fitting of the initial data it is difficult to see why (26) is better in practice.

4. Practical application of finite element methods in forecasting problems

4.1. General considerations

A forecasting model describes the propagation of waves, both gravity waves and Rossby waves. When a wave is propagated through a non-uniform medium internal reflections and refractions take place. Thus when constructing a numerical model to describe wave propagation we want the computational "medium" to be uniform. In the finite element case this rules out the higher order schemes where some nodal values are values at vertices of elements, some are values at midpoints of element sides and some are derivatives. It also rules out schemes on irregular grids. The error analysis of section 3 bears this out; schemes such as the linear scheme where all nodes are equivalent give much greater accuracy. Many of the error terms which vanish do so by symmetry. Higher order finite element schemes are advantageous for steady-state problems where the differences between the types of node do not matter.

When forecasting for a limited area it is natural to use a regular rectangular mesh. However, when forecasting for the hemisphere it is impossible to use a spatially regular mesh, though one can use a regular mesh in latitude and longitude (see chapter 2). It is possible to generate a quasi-regular grid on a polyhedron, usually an icosahedron. This has been done in finite difference models by Sadourny (1972) and Williamson (1970). The triangular faces of the icosahedron are divided into equilateral triangles and the finite element interpolation based on these (Fig 10). On such a grid the finite element method gives a fully two-dimensional scheme. For instance, consider a patch of elements as shown in Fig. 11. The scheme for $\omega = \partial u/\partial x$ using linear elements is

$$\frac{1}{12}(\omega_1 + \omega_2 + \omega_3 + \omega_{45} + \omega_6 + \omega_7) + \tfrac{1}{2}\omega_4 = \frac{1}{60x}\left((u_2 - u_1) + 2(u_5 - u_3) + (u_7 - u_6)\right) \qquad (41)$$

This allows for the y variation of $\partial u/\partial x$. This type of grid was used quite successfully by Cullen (1974b).

Another problem concerns the staggering of the mesh often used in finite difference schemes (Vol.I chapter 4). This is difficult in a finite element scheme because we need to define different elements for each variable and the overlapping would lead to awkward schemes if the Galerkin method were used. A scheme rather similar in effect is shown in Fig 12. All scalar quantities such as pressure are defined to be linear on the triangles, all vectors such as velocities are constant within the triangles. This means that a separate co-ordinate system can be used for each triangle. There is therefore no difficulty with the pole on a sphere. Quantities such as divergence are scalars and are evaluated at the vertices. In order to calculate $\omega = \nabla . \underline{u}$ with $\underline{u}$ constant on the triangles set

$$\int \omega \chi_n = \int \nabla . u \chi_n \qquad (42)$$

where the χ_n are linear basis functions. Then we can integrate the right hand side by parts to give $-\int u . \nabla \chi_n$ which can be calculated on each triangle if u is constant, as $\nabla \chi_n$ is also constant on triangles.

It is of course possible to use vorticity and divergence, or velocity potential and stream function as variables instead of velocity components. This means that second order derivatives appear in the equations and Poisson equations have to be solved. This approach is parallel to that used in spectral models (see chapter 3) and has been used in finite elements by Staniforth and Mitchell (1977). Using linear elements the scheme obtained for the Poisson equation is very similar to the finite difference scheme and can be inverted by the same techniques. However the algorithm is only second order accurate as is the usual finite difference Laplacian.

Semi-implicit time differencing can be used in finite element models exactly as in other models. Finite elements could be used for vertical differencing though this has not yet been done, global polynomial expansions being preferred.

4.2 Programming considerations

The major disadvantage of Galerkin methods like (16) and (22) is the implicit nature of the schemes. The right hand sides will be cheaper to calculate on a regular grid because most of the coefficients will be equal and the terms can be grouped as in a Fast Fourier Transform. We still have to solve equations of the form

$$\tfrac{1}{6}\omega_{n-1} + \tfrac{2}{3}\omega_n + \tfrac{1}{6}\omega_{n+1} = R_n \qquad (43)$$

or more complicated ones, like (41), in two dimensions. (43) is tridiagonal and thus easy to solve. (41) can be solved either by approximate direct inversion or by relaxation using a similar procedure to the standard relaxation method for Poisson's equation. Direct inversion of (43) gives

$$\omega_n = 1.732\,R_n - 0.4638\,(R_{n-1} + R_{n+1}) + 0.1244\,(R_{n-2} + R_{n+2})$$
$$-0.0333\,(R_{n-3} + R_{n+3}) + 0.00899\,(R_{n-4} + R_{n+4}) + \ldots$$

Therefore within 1% this series can be truncated at the terms in $R_{n\pm3}$. This approach was used by Cullen (1973). When using the relaxation method for (41) we have to under-relax to accelerate convergence. The iteration need only be continued until the iteration error is less than the error in calculating R_n . Taking (43) as a simpler example, use as first guess $\omega_n = R_n$. This is exact if w is a constant. It is least accurate if w varies rapidly from point to point, which is when R_n will also be least accurate. In practice 3 iterations with a coefficient of $\frac{2}{3}$ are sufficient.

Results using finite elements can be found in the literature (e.g. Carson and Cullen (1977), Staniforth and Mitchell (1977), Wang et al (1972)). At present the results are similar to those from finite difference models with rather more points. However, the finite element models tend to be more expensive in computer time.

Figure Captions

Fig.1 The simplest linear finite element approximation.

Fig.2 Some other simple finite elements:

A Cubic Hermite

B Two dimensional linear element

Fig.3 Basis functions for simple finite element approximations.

A Linear element

B Cubic Hermite element, basis function for nodal value.

C Cubic Hermite element, basis function for nodal derivative

Fig.4 Calculation of the square of a piecewise linear function

A Original function u

B u^2

C u^2, using point collocation

D u^2, using Galerkin

Fig.5 Alternative schemes for non-linear advection

A Function u

B Function v

C $\partial v/\partial x$, (i), and its best fit by a linear function (ii))

D $u\partial v/\partial x$ using (i) multiplication of curve A by curve C(i) and,(ii),the best fit to it by a linear function. (Equation (25))

E $u\partial v/\partial x$ using (i) multiplication of curve A by curve C(ii) and,(ii),the best fit to it by a linear function. (Equation (26)).

Fig.6 Quadratic basis functions

A Endpoint function

B Midpoint function

Fig.7 Error analysis for evolutionary problem

Fig.8 Different linear finite element approximations to a wave

A Best least squares fit

B Interpolated values

Fig.9 Relative phase speeds of second and fourth order (A and B) and linear finite element scheme (C). Exact solution represented as unity.

Fig.10 Grid of triangles on sphere based on icosahedron.

Fig.11 Local patch of elements to define scheme (41).

Fig.12 Finite element scheme with scalars and vectors treated separately.

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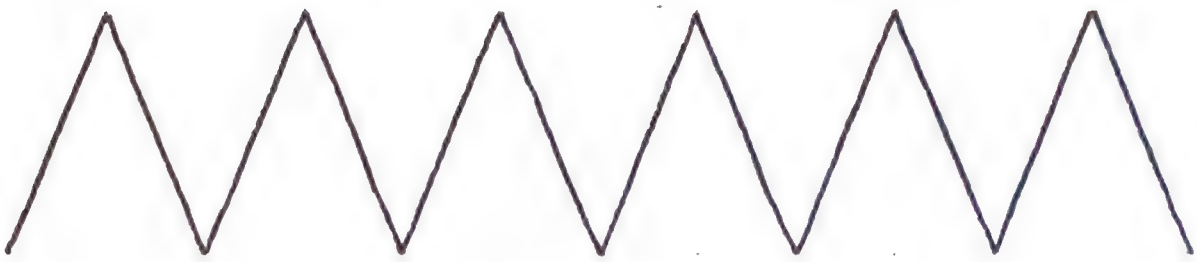
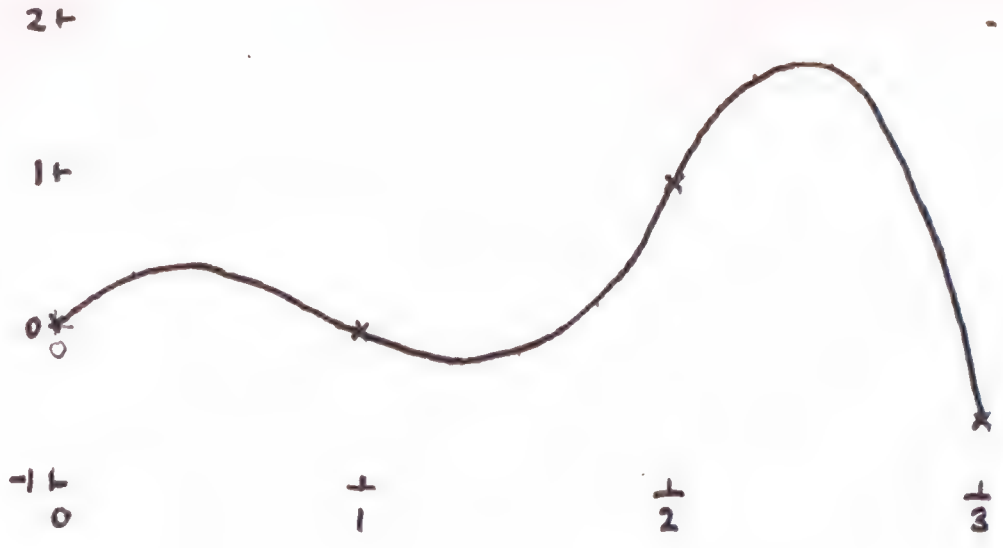


FIG. 1



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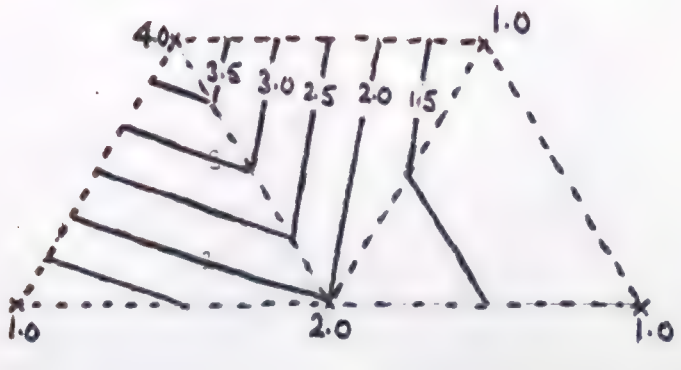
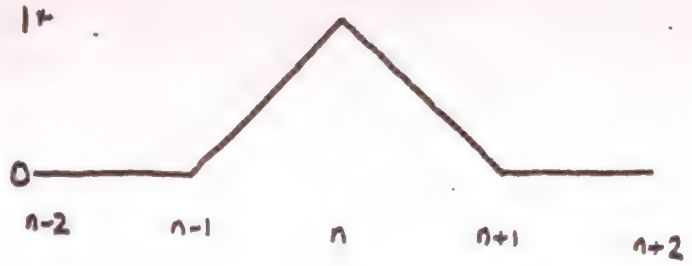


(b)

FIG 2



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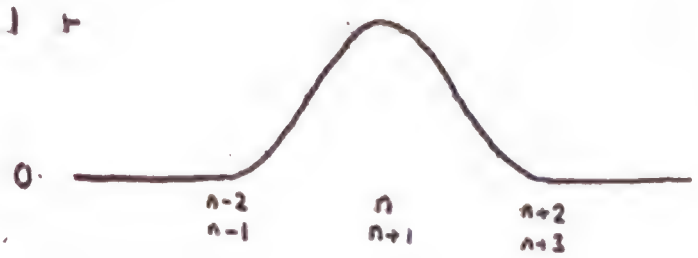

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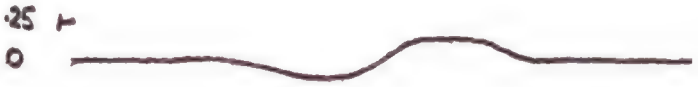

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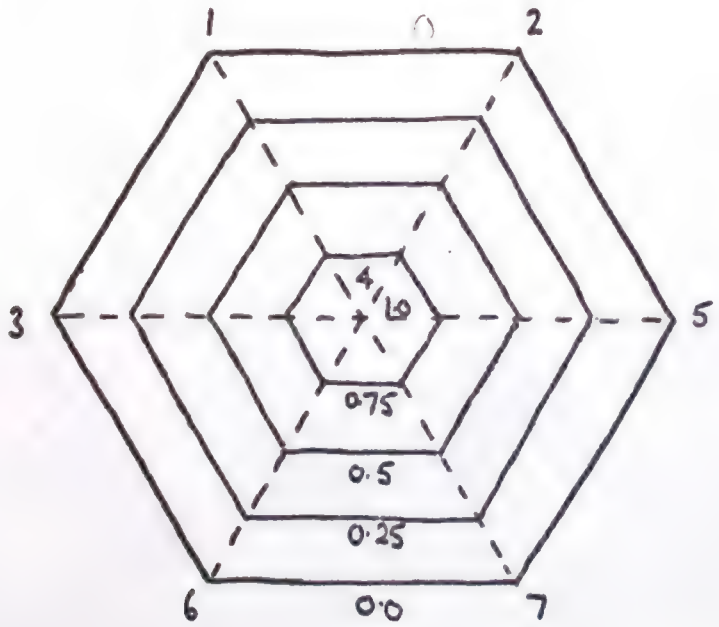


(d)

FIG 3

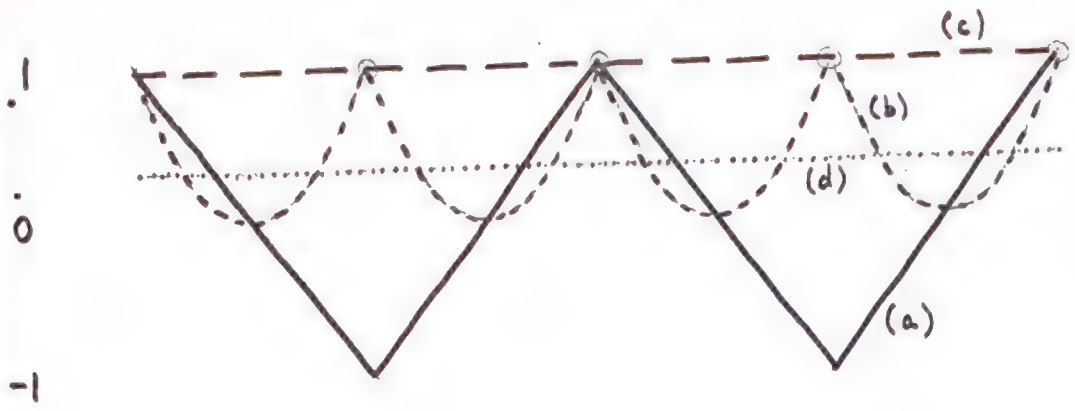


FIG 4

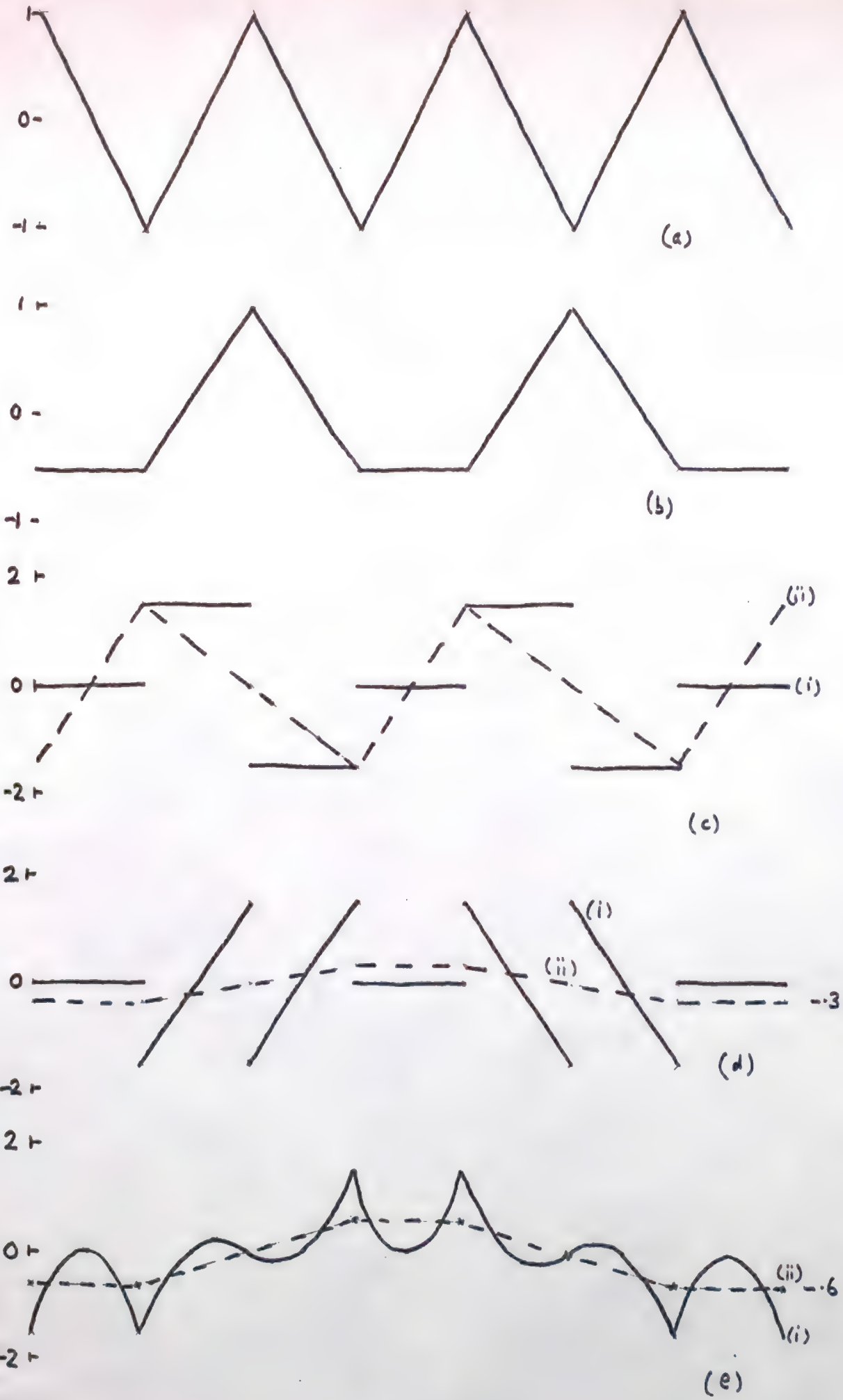
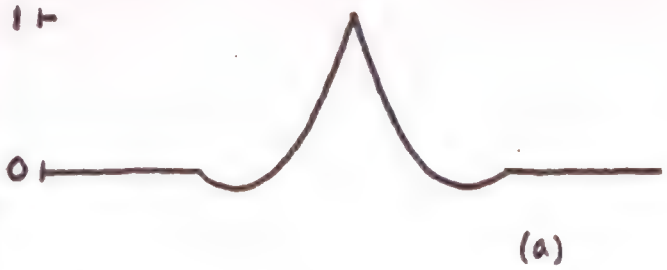


FIG 5



(a)

(b)

FIG 6

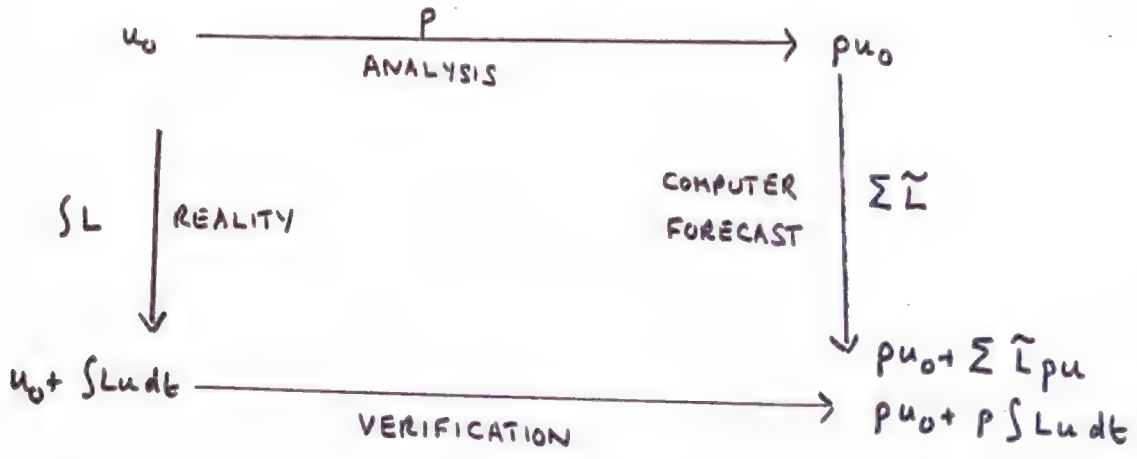

u_0
P
ANALYSIS
pu_0
$\int L$
REALITY
COMPUTER
FORECAST
$\Sigma \tilde{L}$
$u_0 + \int L u \, dt$
$pu_0 + \Sigma \tilde{L} pu$
VERIFICATION
$pu_0 + p \int L u \, dt$

FIG. 7

Fig 8

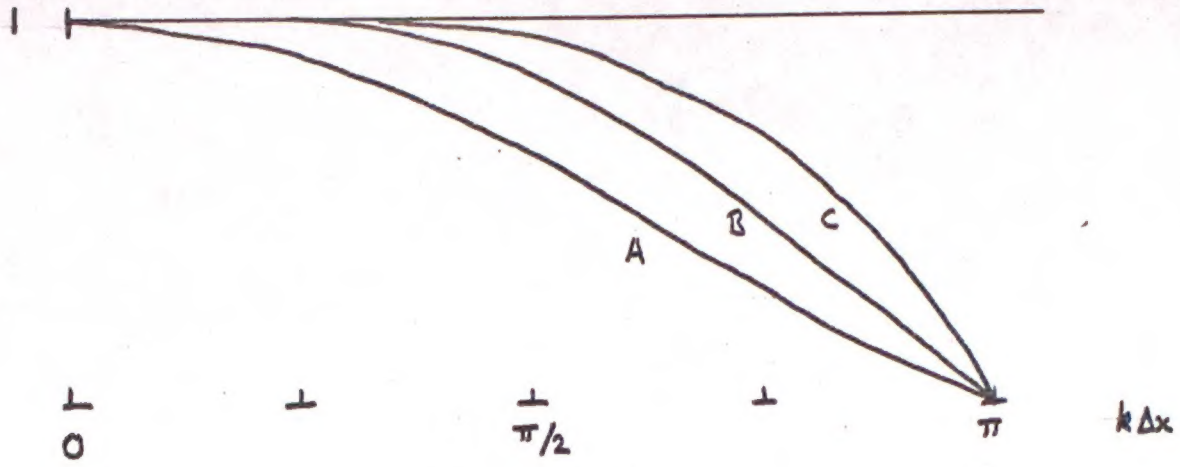
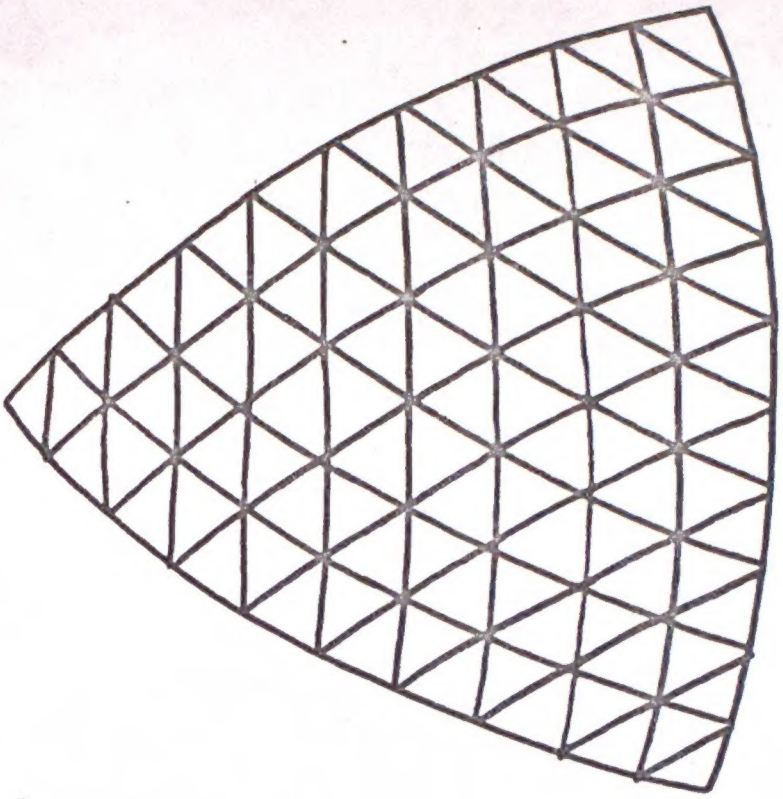


Fig 9



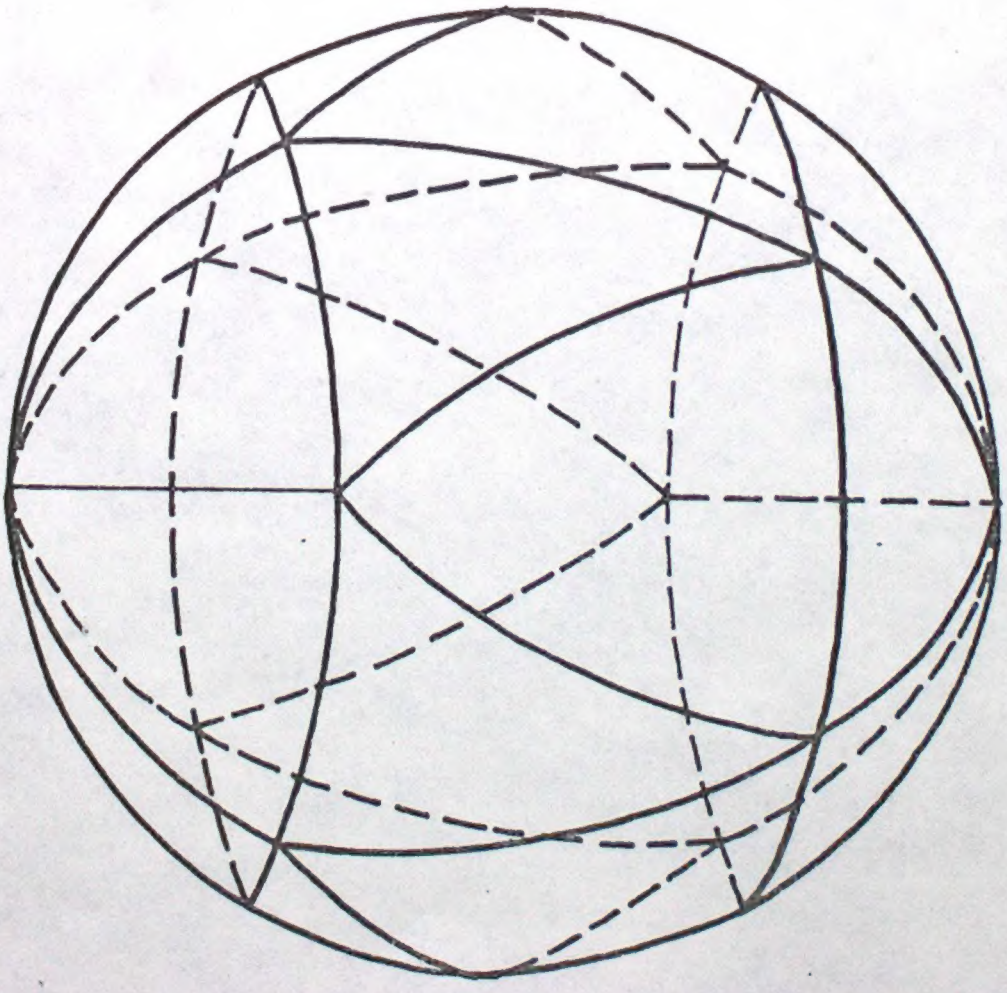


FIG. 10

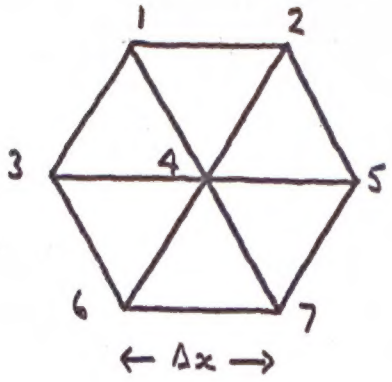


FIG 11

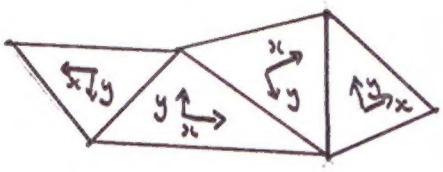


FIG. 12.